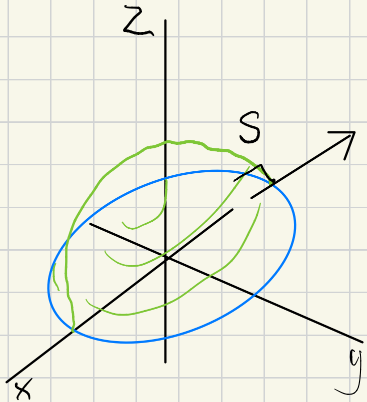
Pre-class Warm-up!!!

Would you choose to use Stokes' theorem to do the following problem?

Let F be the vector field
F(x,y,z) = (x + y^3z, sin(x) + z, y).
Compute the flux of F across the upper unit hemisphere S, that is
∬_S F · dS



a. Yes

b. No

The issue: $F \neq \nabla \times G$ for any G

because $\nabla \cdot F = 1+0+0 = 1 \neq 0$

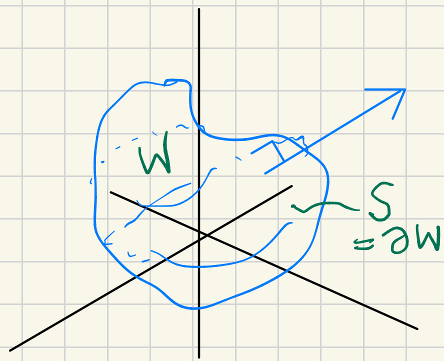
Stokes says

$$\iint_S \nabla \times G \cdot d\underline{S} = \int_{\partial S} G \cdot d\underline{s}$$

Section 8.4: Gauss's Theorem

We learn:

- Gauss's theorem is similar to Green's theorem and Stokes' theorem, except it works one dimension higher.
- How the surface of a solid region must be oriented for Gauss's theorem
- The use of Gauss's theorem in computing the flux of a vector field across a surface.
- The proof looks very like the proof of Green's theorem
- It is also called the Divergence Theorem.



Gauss's Theorem.
Let W be a solid region of R^3 bounded by a closed surface S = ∂W oriented so that the unit normal points to the outside of the W.
If F : R^3 -> R^3 is a vector field we have

$$\iiint_W \nabla \cdot F \, dV = \iint_{S=\partial W} F \cdot d\underline{S}$$

What do closed surface and ∂W mean?

"Closed" means bounded, with empty boundary $\partial S = \emptyset$

W is the 3-D region inside a closed surface.

Example. (Like most HW questions)

Find the flux of F across S where S is the unit sphere with outward pointing normal, and

F(x,y,z) = $\left(x + e^{\cos y}\sqrt{1+z^2},\ y + \frac{x}{1+z^2},\ 2z - 3y^3\right)$

Solution. Let W be the solid unit ball so $\partial W = S$. By Gauss:

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_W \nabla \cdot \mathbf{F}\, dV$$

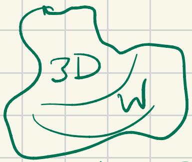
$$= \iiint_W 1 + 1 + 2\, dV = 4 \iiint_W dV$$

$$= 4 \text{ volume of } W = \frac{16\pi}{3}. \quad \square$$

(x + e^cos(y) √1+z^2, y + x/(1+z^2), 2z-3y^3)

Comparison with Stokes' theorem.

Gauss:

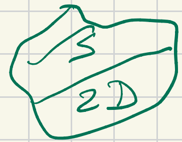


$S = \partial W$ is 2D

$$\iiint_W \nabla \cdots$$

$$= \iint_{\partial W} \cdots$$

Stokes



$C = \partial S$ is 1-D

$$\iint_S \nabla \cdots$$

$$= \int_{\partial S} \cdots$$

If Div F = 0 and S is closed we can deduce that ∬_S F·dS = 0 in two ways.

1. Use Stokes: $\operatorname{Div} F = 0 \Rightarrow F = \nabla \times G$ for some G

$$\iint_S F \cdot dS = \iint_S \nabla \times G \cdot d\underline{S}$$

$$= \int_{\partial S} G \cdot d\underline{s} = 0$$ because ∂S is empty.

2. Use Gauss: Let W be bounded by S

$$\iint_S F \cdot dS = \iiint_W \nabla \cdot F \, dV$$

$$= \iiint_W 0 \, dV = 0.$$

Example (closing a surface)

Let F(x,y,z) =
(x + e^cos(y) √1+z^2, y + x/(1+z^2),
2z-3y^3)

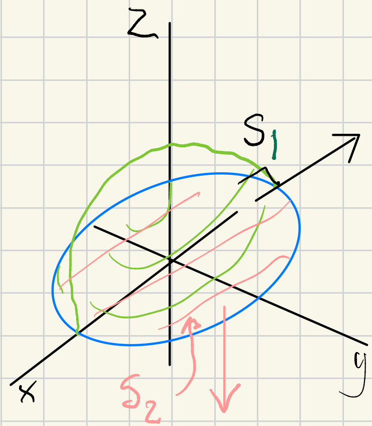
Find the flux of F across the upper half of the unit sphere, oriented with an outward pointing normal.

Solution using Gauss: Let W be the upper half of the unit ball

$$\partial W = S_1 \cup S_2$$

$$\iiint_W \nabla \cdot F \, dV$$

$$= \iint_{S_1} F \cdot d\underline{S} + \iint_{S_2} F \cdot d\underline{S}$$



$$\iint_{S_1} F \cdot d\underline{S} = \iiint_W \nabla \cdot F \, dV - \iint_{S_2} F \cdot d\underline{S}$$

$$\iiint_W (1+1+2) \, dV = 4 \cdot \frac{2}{3}\pi$$

$\iint_{S_2} F \cdot d\underline{S}$: on S_2 the unit normal is $(0,0,-1)$

$$F \cdot (0,0,-1) = -2z + 3y^3.$$

We get $\displaystyle\iint_{\substack{\text{unit} \\ \text{circle in } xy \text{ plane}}} -2z + 3y^3 \, dx\, dy$

$= 0 + 0 \quad (3y^3 \text{ is odd, } 2z = 0)$

$$\iint_{S_1} F \cdot d\underline{S} = \frac{8\pi}{3} \qquad \square$$

Example. (Solid regions with holes in them)
Theorem 10 in the book:

The flux of $(x,y,z) / r^3$ across the surface of an 'elementary symmetric region' M is
4π if $(0,0,0)$ lies in M
0 if $(0,0,0)$ does not lie in M.